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ROOTS OF THE AFFINE CREMONA GROUP

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Abstract. Let $\mathbf{k}^{[n]} = \mathbf{k}[x_1, \ldots, x_n]$ be the polynomial algebra in n variables and let $\mathbb{A}^n = \operatorname{Spec} \mathbf{k}^{[n]}$. In this note we show that the root vectors of the affine Cremona group $\operatorname{Aut}(\mathbb{A}^n)$ with respect to the diagonal torus are exactly the locally nilpotent derivations $\mathbf{x}^\alpha \frac{\partial}{\partial x_i}$, where $\mathbf{x}^\alpha$ is any monomial not depending on x_i. This answers a question due to Popov.

Introduction

Letting $\mathbf{k}$ be an algebraically closed field of characteristic zero, we let $\mathbf{k}^{[n]} = \mathbf{k}[x_1, \ldots, x_n]$ be the polynomial algebra in n variables, and $\mathbb{A}^n = \operatorname{Spec} \mathbf{k}^{[n]}$ be the affine space. The Cremona group $\operatorname{Aut}(\mathbb{A}^n)$ is the group of automorphisms of $\mathbb{A}^n$, or equivalently, the group of $\mathbf{k}$-automorphisms of $\mathbf{k}^{[n]}$. We define $\operatorname{Aut}^*(\mathbb{A}^n)$ as the subgroup of volume preserving automorphisms i.e.,

$$\operatorname{Aut}^*(\mathbb{A}^n) = \left\{ \gamma \in \operatorname{Aut}(\mathbb{A}^n) \mid \det \left(\frac{\partial}{\partial x_i} \gamma(x_j) \right)_{i,j} = 1 \right\} .$$

The groups $\operatorname{Aut}(\mathbb{A}^n)$ and $\operatorname{Aut}^*(\mathbb{A}^n)$ are infinite dimensional algebraic groups [Sha66, Kam79].

It follows from [BB66, BB67] that the maximal dimension of an algebraic torus contained in $\operatorname{Aut}^*(\mathbb{A}^n)$ is $n-1$. Moreover, every algebraic torus of dimension $n-1$ contained in $\operatorname{Aut}^*(\mathbb{A}^n)$ is conjugated to the diagonal torus

$$\mathbf{T} = \{\gamma = \operatorname{diag}(\gamma_1, \ldots, \gamma_n) \in \operatorname{Aut}^*(\mathbb{A}^n) \mid \gamma_1 \cdots \gamma_n = 1\} .$$

A $\mathbf{k}$-derivation ∂ on an algebra A is called locally nilpotent (LND for short) if for every $a \in A$ there exists $k \in \mathbb{Z}_{\geq 0}$ such that $\partial^k(a) = 0$. If $\partial : \mathbf{k}^{[n]} \to \mathbf{k}^{[n]}$ is an LND on the polynomial algebra, then $\exp(t\partial) \in \operatorname{Aut}^*(\mathbb{A}^n)$, for all $t \in \mathbf{k}$ [Fre06]. Hence, ∂ belongs to the Lie algebra $\operatorname{Lie}(\operatorname{Aut}^*(\mathbb{A}^n))$.

In analogy with the notion of root from the theory of algebraic groups [Spr98], Popov introduced the following definition. A non-zero locally nilpotent derivation ∂ on $\mathbf{k}^{[n]}$ is called a root vector of $\operatorname{Aut}^*(\mathbb{A}^n)$ with respect to the diagonal torus $\mathbf{T}$ if there exists a character χ of $\mathbf{T}$ such that

$$\gamma \circ \partial \circ \gamma^{-1} = \chi(\gamma) \cdot \partial, \quad \text{for all} \quad \gamma \in \mathbf{T} .$$

The character χ is called the root of $\operatorname{Aut}^*(\mathbb{A}^n)$ with respect to $\mathbf{T}$ corresponding to ∂.

Letting $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, we let $\mathbf{x}^{\boldsymbol{\alpha}}$ be the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. In this note we apply the results in [Lie10] to prove the following theorem. This answers a question due to Popov [FR05].

Theorem 1. *If ∂ is a root vectors of $\operatorname{Aut}^*(\mathbb{A}^n)$ with respect to the diagonal torus $\mathbf{T}$, then*

$$\partial = \lambda \cdot \mathbf{x}^{\boldsymbol{\alpha}} \cdot \frac{\partial}{\partial x_i} ,$$

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for some $\lambda \in \mathbf{k}^*$, some $i \in \{1, \ldots, n\}$, and some $\boldsymbol{\alpha} \in \mathbb{Z}^n_{\geq 0}$ such that $\alpha_i = 0$. The corresponding root is the character

$$\chi : \mathbf{T} \to \mathbf{k}^*, \quad \gamma = \operatorname{diag}(\gamma_1, \ldots, \gamma_n) \mapsto \gamma_i^{-1} \prod_{j=1}^{n} \gamma_j^{\alpha_j}\,.$$

1. Proof of the Theorem

It is well known that the set $\operatorname{Char}(\mathbf{T})$ of characters of $\mathbf{T}$ forms a lattice isomorphic to $M = \mathbb{Z}^{n-1}$. It is customary to fix an isomorphism $M \simeq \operatorname{Char}(\mathbf{T})$ and to denote the character corresponding to m by χ^m. The natural $\mathbf{T}$-action on $\mathbb{A}^n$ gives rise to an M-grading on $\mathbf{k}^{[n]}$ given by

$$\mathbf{k}^{[n]} = \bigoplus_{m \in M} B_m, \quad \text{where} \quad B_m = \left\{ f \in \mathbf{k}^{[n]} \mid \gamma(f) = \chi^m(\gamma) f,\, \forall \gamma \in \mathbf{T} \right\}.$$

An LND ∂ on $\mathbf{k}^{[n]}$ is called homogeneous if it send homogeneous elements into homogeneous elements. Let $f \in \mathbf{k}^{[n]} \setminus \ker \partial$ homogeneous. We define the degree of ∂ as $\deg \partial = \deg(\partial(f)) - \deg(f) \in M$ [Lie10, Section 1.2].

Lemma 2. *An LND on $\mathbf{k}^{[n]}$ is a root vectors of $\operatorname{Aut}^*(\mathbb{A}^n)$ with respect to the diagonal torus $\mathbf{T}$ if and only if ∂ is homogeneous with respect to the M-grading on $\mathbf{k}^{[n]}$ given by $\mathbf{T}$. Furthermore, the corresponding root is the character $\chi^{\deg \partial}$.*

Proof. Let ∂ be a root vector of $\operatorname{Aut}^*(\mathbb{A}^n)$ with root χ^e, so that $\partial = \chi^{-e}(\gamma) \cdot \gamma \circ \partial \circ \gamma^{-1}$, $\forall \gamma \in \mathbf{T}$. We consider a homogeneous element $f \in B_{m'}$ and we let $\partial(f) = \sum_{m \in M} g_m$, where g_m is homogeneous, so that

$$\sum_{m \in M} g_m = \partial(f) = \chi^{-e}(\gamma) \cdot \gamma \circ \partial \circ \gamma^{-1}(f) = \chi^{-e-m'}(\gamma) \sum_{m \in M} \chi^m(\gamma) \cdot g_m, \quad \forall \gamma \in \mathbf{T}\,.$$

This equality holds if and only if $g_m = 0$ for all but one $m \in M$ i.e., if ∂ is homogeneous. In this case, $\partial(f) = g_m = \chi^{-e-m'+m}(\gamma) \cdot \partial(f)$, and so $e = m - m' = \deg(\partial(f)) - \deg(f) = \deg \partial$. ☐

In [AH06], a combinatorial description of a normal affine M-graded domain A is given in terms of polyhedral divisors, and in [Lie10] a description of the homogeneous LNDs on A is given in terms of these combinatorial data in the case where $\operatorname{tr.deg} A = \operatorname{rank} M + 1$. In the following we apply these results to compute the homogeneous LNDs on the M-graded algebra $\mathbf{k}^{[n]}$. First, we give a short presentation of the combinatorial description in [AH06] in the case where $\operatorname{tr.deg} A = \operatorname{rank} M + 1$. For a more detailed treatment see [Lie10, Section 1.1].

Let N be the dual lattice of M. The combinatorial description in [AH06] deals with the following data: A pointed polyhedral cone $\sigma \in N_{\mathbb{Q}} := N \otimes \mathbb{Q}$ dual to the weight cone $\sigma^\vee \subseteq M_{\mathbb{Q}} := M \otimes \mathbb{Q}$ of the M-grading; a smooth curve C; and a divisor $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ on C whose coefficients Δ_z are polyhedra in $N_{\mathbb{Q}}$ having tail cone σ. Furthermore, if C is projective we ask for the polyhedron $\sum_{z \in C} \Delta_z$ to be a proper subset of σ. For every $m \in \sigma^\vee \cap M$ the evaluation of $\mathfrak{D}$ at m is the $\mathbb{Q}$-divisor given by

$$\mathfrak{D}(m) = \sum_{z \in C} \min_{p \in \Delta_z} p(m)\,.$$

We define the M-graded algebra

$$A[\mathfrak{D}] = \bigoplus_{m \in \sigma^\vee \cap M} A_m \chi^m, \quad \text{where} \quad A_m = H^0(C, \mathcal{O}_C(\mathfrak{D}(m)))\,, \tag{1}$$

and χ^m is the corresponding character of the torus $\operatorname{Spec} \mathbf{k}[M]$ seen as a rational function on $\operatorname{Spec} A$ via the embedding $\operatorname{Frac} \mathbf{k}[M] \hookrightarrow \operatorname{Frac} A[\mathfrak{D}] = \operatorname{Frac} \mathbf{k}(C)[M]$.

It follows from [AH06] that $A[\mathfrak{D}]$ is an normal affine domain and that every normal affine M-graded domain A with $\operatorname{tr.deg} A = \operatorname{rank} M+1$ is equivariantly isomorphic to $A[\mathfrak{D}]$ for some polyhedral divisor on a smooth curve, see also [Lie10, Theorem 1.4].

We turn back now to our particular case where we deal with the polynomial algebra $\mathbf{k}^{[n]}$ graded by $\operatorname{Char}(\mathbf{T})$. First, we need to fix an isomorphism $M \simeq \operatorname{Char}(\mathbf{T})$. Let $\{\mu_1, \ldots, \mu_{n-1}\}$ be the canonical basis of M. We define the isomorphism $\mu_i \mapsto \chi^{\mu_i}$, $i \in \{1, \ldots, n-1\}$, where

$$\chi^{\mu_i} : \mathbf{T} \to \mathbf{k}^*, \quad \gamma = \operatorname{diag}(\gamma_1, \ldots, \gamma_n) \mapsto \gamma_i \,.$$

Since for every $\gamma \in \mathbf{T}$ we have $\gamma_n = \gamma_1^{-1} \cdots \gamma_{n-1}^{-1}$, the character mapping $\operatorname{diag}(\gamma_1, \ldots, \gamma_n) \mapsto \gamma_n$ is given by $\chi^{-\mathbb{1}}$, where $\mathbb{1} := \mu_1 + \ldots + \mu_{n-1}$. Under this isomorphism, the algebra $\mathbf{k}^{[n]}$ is graded by M via $\deg x_i = \mu_i$, for all $i \in \{1, \ldots, n-1\}$ and $\deg x_n = -\mathbb{1}$.

Let now $\{\nu_1, \ldots, \nu_{n-1}\}$ be a basis of N dual to the basis $\{\mu_1, \ldots, \mu_{n-1}\}$ of M. We also let Δ be the standard $(n-1)$-simplex in $N_{\mathbb{Q}}$ i.e., the convex hull of the set $\{\nu_1, \ldots, \nu_{n-1}, \bar{0}\}$.

Lemma 3. *The M-graded algebra $\mathbf{k}^{[n]}$ is equivariantly isomorphic to $A[\mathfrak{D}]$, where $\mathfrak{D}$ is the polyhedral divisor $\mathfrak{D} = \Delta \cdot [0]$ on $\mathbb{A}^1$.*

Proof. By [AH06], the M-graded algebra $\mathbf{k}^{[n]}$ is isomorphic to $A[\mathfrak{D}]$ for some polyhedral divisor $\mathfrak{D}$ on a smooth curve C. Since the weight cone $\sigma^\vee$ of $\mathbf{k}^{[n]}$ is $M_{\mathbb{Q}}$, the coefficients of $\mathfrak{D}$ are just bounded polyhedra in $N_{\mathbb{Q}}$.

Since $\mathbb{A}^n$ is a toric variety and the torus $\mathbf{T}$ is a subtorus of the big torus, we can apply the method in [AH06, Section 11]. In particular, C is a toric curve. Thus $C = \mathbb{A}^1$ or $C = \mathbb{P}^1$. Furthermore, the graded piece $B_{\bar{0}} \supsetneq \mathbf{k}$ and so C is not projective by (1). Hence $C = \mathbb{A}^1$.

The only divisor in $\mathbb{A}^1$ invariant by the big torus is $[0]$, so $\mathfrak{D} = \Delta \cdot [0]$ for some bounded polyhedron Δ in $N_{\mathbb{Q}}$. Finally, applying the second equation in [AH06, Section 11], a routine computation shows that Δ can be chosen as the standard $(n-1)$-simplex. □

Remark 4. Letting $\mathbb{A}^1 = \operatorname{Spec} \mathbf{k}[t]$, it is easily seen that the isomorphism $\mathbf{k}^{[n]} \simeq A[\mathfrak{D}]$ is given by $x_i = \chi^{\mu_i}$, for all $i \in \{1, \ldots, n-1\}$, and $x_n = t\chi^{-\mathbb{1}}$.

In [Lie10] the homogeneous LNDs on an normal affine M-graded domain are classified into 2 types: fiber type and horizontal type. In the case where the weight cone is $M_{\mathbb{Q}}$, there are no LNDs of fiber type. Thus, $\mathbf{k}^{[n]}$ only admits homogeneous LNDs of horizontal type. The homogeneous LNDs of horizontal type are described in [Lie10, Theorem 3.28]. In the following, we specialize this result to the particular case of $A[\mathfrak{D}] \simeq \mathbf{k}^{[n]}$.

Let $v_i = \nu_i$, $i \in \{1, \ldots, n-1\}$ and $v_n = \bar{0}$, so that $\{v_1, \ldots, v_n\}$ is the set of vertices of Δ. For every $\lambda \in \mathbf{k}^*$, $i \in \{1, \ldots, n\}$, and $e \in M$ we let $\partial_{\lambda,i,e} : \operatorname{Frac} A[\mathfrak{D}] \to \operatorname{Frac} A[\mathfrak{D}]$ be the derivation given by

$$\partial_{\lambda,i,e}(t^r \cdot \chi^m) = \lambda(r + v_i(m)) \cdot t^{r - v_i(e) - 1} \cdot \chi^{m+e}, \quad \forall (m, r) \in M \times \mathbb{Z}\,.$$

Lemma 5 ([Lie10, Theorem 3.28])**.** *If ∂ is a non-zero homogeneous LND of $A[\mathfrak{D}]$, then $\partial = \partial_{\lambda,i,e}|_{A[\mathfrak{D}]}$ for some $\lambda \in \mathbf{k}^*$, some $i \in \{1, \ldots, n\}$, and some $e \in M$ satisfying $v_j(e) \geq v_i(e)+1$, $\forall j \neq i$. Furthermore, e is the degree* $\deg \partial$.

Proof of Theorem 1. By Lemma 2 the root vectors of $\mathbf{k}^{[n]}$ correspond to the homogeneous LNDs in the M-graded algebra $\mathbf{k}^{[n]}$. But the homogeneous LNDs on $A[\mathfrak{D}] \simeq \mathbf{k}^{[n]}$ are given in Lemma 5, so we only need to translate the homogeneous LND $\partial = \partial_{\lambda,i,e}|_{A[\mathfrak{D}]}$ in Lemma 5 in terms of the explicit isomorphism given in Remark 4.

Let $e = (e_1, \ldots, e_{n-1})$ and $i \in \{1, \ldots, n-1\}$, so that $v_i = \mu_i$. The condition $v_j(e) \geq v_i(e)+1$ yields $e_i \leq -1$ and $e_j \geq e_i + 1$, $\forall j \neq i$. Furthermore, $\partial(x_k) = \partial(\chi^{\mu_k}) = 0$, for all $k \neq i$, $k \in \{1, \ldots, n-1\}$,

$\partial(x_n)=\partial(t\chi^{-\mathbb{1}})=0$, and
$$\partial(x_i)=\partial(\chi^{\mu_i})=\lambda t^{-e_i-1}\chi^{e+\mu_i}=\lambda t^{-e_i-1}\chi^{(e_i+1)\mathbb{1}}\chi^{e+\mu_i-(e_i+1)\mathbb{1}}=\lambda\mathbf{x}^{\boldsymbol{\alpha}}\,,$$
where $\alpha_i=0$, $\alpha_n=-e_i-1\geq 0$, and $\alpha_k=e_k-e_i-1\geq 0$, for all $k\neq i$, $k\in\{1,\ldots,n-1\}$. Hence, $\partial=\lambda\cdot\mathbf{x}^{\boldsymbol{\alpha}}\cdot\frac{\partial}{\partial x_i}$, for some $\lambda\in\mathbf{k}^*$, some $i\in\{1,\ldots,n\}$, and some $\boldsymbol{\alpha}\in\mathbb{Z}^n_{\geq 0}$ such that $\alpha_i=0$.

Let now $e=(e_1,\ldots,e_{n-1})$ and $i=n$, so that $v_n=0$. The condition $v_j(e)\geq v_n(e)+1$ yields $e_j\geq 1$, $\forall j\in\{1,\ldots,n-1\}$. Furthermore, $\partial(x_k)=\partial(\chi^{\mu_k})=0$, $k\in\{1,\ldots,n-1\}$, and
$$\partial(x_n)=\partial(t\chi^{-\mathbb{1}})=\lambda\chi^{e-\mathbb{1}}=\lambda\mathbf{x}^{\boldsymbol{\alpha}}\,,$$
where $\alpha_n=0$, and $\alpha_k=e_k-1\geq 0$, for all $k\in\{1,\ldots,n-1\}$. Hence, $\partial=\lambda\cdot\mathbf{x}^{\boldsymbol{\alpha}}\cdot\frac{\partial}{\partial x_n}$, for some $\lambda\in\mathbf{k}^*$, some $i\in\{1,\ldots,n\}$, and some $\boldsymbol{\alpha}\in\mathbb{Z}^n_{\geq 0}$ such that $\alpha_n=0$.

The last assertion of the theorem follows easily from the fact that the root corresponding to the homogeneous LND ∂ is the character $\chi^{\deg\partial}$. $\square$

Finally, we describe the characters that appear as a root of $\mathrm{Aut}^*(\mathbb{A}^n)$.

Corollary 6. *The character $\chi\in\mathrm{Char}(\mathbf{T})$ given by $\mathrm{diag}(\gamma_1,\ldots,\gamma_n)\mapsto\gamma_1^{\beta_1}\cdots\gamma_n^{\beta_n}$ is a root of $\mathrm{Aut}^*(\mathbb{A}^n)$ with respect to the diagonal torus $\mathbf{T}$ if and only if the minimum of the set $\{\beta_1,\ldots,\beta_n\}$ is achieved by one and only one of the β_i.*

Proof. By Theorem 1, the roots of $\mathrm{Aut}^*(\mathbb{A}^n)$ are the characters $\mathrm{diag}(\gamma_1,\ldots,\gamma_n)\mapsto\gamma_1^{\beta_1}\cdots\gamma_n^{\beta_n}$, where $\beta_i=-1$ for some $i\in\{1,\ldots,n\}$ and $\beta_j\geq 0$ $\forall j\neq i$. The corollary follows from the fact that $\gamma_1\cdots\gamma_n=1$. $\square$

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